PHIL 4310: Advanced Logic
Spring 2026
Homework 6

This homework is due Mon, April 27th.
Finish reading Chapter 4 of *Logic for Philosophy* (at least through section 4.6).

Homework questions:

1) Do exercise 4.6h (page 194).
2) Do exercise 4.8b (page 196).
3) Do exercise 4.9c (page 197).
4) Do exercise 4.11a (page 199).
5) Do exercise 4.13b (page 200).
6) Do exercise 4.15a (page 202).
7) Do exercise 4.17 (page 203).
8) Do exercise 4.22 (page 208).

9) Show that the K◊ rule is redundant (page 185). (Hint: Use the modal negation rules to move between □ and ◊ sentences).

Correspondence of axioms and frames:

Recall that a frame is a non-empty set of worlds with an accessibility relation between worlds. So it *part* of a model that doesn't have any truth value assignments to formulas. A formula is said to be *valid* on a frame if will turn out true at every world in that frame regardless of the truth-value assignment to the atomic sentences.

A formula is said to correspond to a condition on frames if it is valid on all and only frames that satisfy that condition. For example, $\Box P \rightarrow P$ corresponds to reflexivity ($\forall w Rww$) because if every world can see itself in a given frame then $\Box P \rightarrow P$ will be true at every world and if any world cannot see itself, then it is possible to construct a model where $\Box P \rightarrow P$ turns out false at a world (pick a world that doesn't see itself, make P false there and make P true everywhere else).

Some of the more commonly studied frame conditions are listed in section 8 here: https://plato.stanford.edu/entries/logic-modal/

One condition that Garson calls "functional" is the frame condition $\forall w \forall v \forall u\ ((Rwv \wedge Rwu) \rightarrow v = u)$. In other words, each world can see at most one world.

10) Prove that $\Diamond P \rightarrow \Box P$ is valid on a frame iff that frame satisfies this "functional" condition. (So first assume every world can see at most one world and prove the formula will be true everywhere and next assume the frame doesn't satisfy the condition and prove there is a countermodel).

11) Another important condition is "convergence":

$\forall w \forall v \forall u\ ((Rwv \wedge Rwu) \rightarrow \exists x(Rvx \wedge Rux))$

This is called "convergence" because if you have a world that can see two possibilities, they will "converge" back together and not stay separate. (Think of time branching and then converging back to a single future).

Prove that $\Diamond\Box P \rightarrow \Box\Diamond P$ corresponds to the convergence frame condition.

HINT:

Here is a sample, very complete answer. $\Box P \rightarrow \Box\Box P$ corresponds to the frame condition $\forall w \forall v \forall u((Rwv \wedge Rvu) \rightarrow Rwu)$. Proof:

Assume that the frame is transitive. Now pick an arbitrary world w in that frame. Assume that $\Box P$ is true at w. That means that P is true at every world that w can see. Now for reductio, assume that $\Box\Box P$ is false at w. That means that there is a world v where Rwv and where $\Box P$ is false at v. That means that there is a world u where Rvu and where P is false at u. So now we have Rwv and Rvu and so by the frame condition (transitivity) we have Rwu. Since Rwu and $\Box P$ is true at w, we now have P is true at u. This is a contradiction. So $\Box\Box P$ must be true at w. So therefore $\Box P \rightarrow \Box\Box P$ is true at w. w was arbitrary so $\Box P \rightarrow \Box\Box P$ must be true at every world in the frame. This was an arbitrary transitive frame, so $\Box P \rightarrow \Box\Box P$ must be valid in every transitive frame.

Now assume that the frame is NOT transitive. That means $\forall w \forall v \forall u((Rwv \wedge Rvu) \rightarrow Rwu)$ is false which means $\exists w \exists v \exists u(Rwv \wedge Rvu \wedge {\sim}Rwu)$. We will now construct a countermodel:

Make P true at every world in the frame except for u. Now since Rwu is false and P is true everywhere except u, $\Box P$ will be true at w. However, we have Rvu and P is false at u, therefore $\Box P$ is false at v. Now since Rwv and $\Box P$ is false at v, we have $\Box\Box P$ is false at w. Therefore $\Box P$ is true at w and $\Box\Box P$ is false at w so $\Box P \rightarrow \Box\Box P$ is false at w. So $\Box P \rightarrow \Box\Box P$ is not valid in this particular frame. This frame was totally arbitrary (except that it didn't satisfy transitivity) so therefore $\Box P \rightarrow \Box\Box P$ must not be valid in any frame that is not transitive.